Math 2010 Week 6

> **Defn** Let $\Omega \subseteq \mathbb{R}^n$ be open, $f: \Omega \to \mathbb{R}$
>
> Let $r \geq 0$. f is called a **C^r function** if all partial derivatives of f up to order r exist and are continuous on Ω
>
> f is called a **C^∞ function** if it is C^r for any $r \geq 0$

eg ① f is C^0 if it is continuous

② $f(x,y)$ is C^2 if

$$f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}$$

exist and are continuous

Examples of C^∞ function

Polynomials, Rational functions,

Exponential, Logarithm, Trigonometric functions

and their sum / difference / product / quotient / compositions

eg. $e^{x^2-y} \sin \frac{x}{y}$

Generalization of Clairaut's thm

> If f is C^r on an open set $\Omega \subseteq \mathbb{R}^n$, then the order of differentiation does not matter for all partial derivatives up to order r.

eg If $f(x,y,z)$ is C^3, then

$$f_{xz} = f_{zx}, \quad f_{xyz} = f_{yzx} = f_{zyx}$$

$$f_{xxy} = f_{xyx} = f_{yxx}$$

Differentiability

1 variable case revisited

$f: \mathbb{R} \to \mathbb{R}$ is differentiable at a if

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists}$$

Multivariable case: $f: \mathbb{R}^n \to \mathbb{R}$, $\vec{a} \in \mathbb{R}^n$

Same definition?

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a})}{\underbrace{\vec{x} - \vec{a}}} \quad \begin{matrix} \leftarrow \mathbb{R} \ \checkmark \\ \leftarrow \mathbb{R}^n \ \times \end{matrix}$$

Doesn't make sense to divide by a vector

Need another way to define differentiability

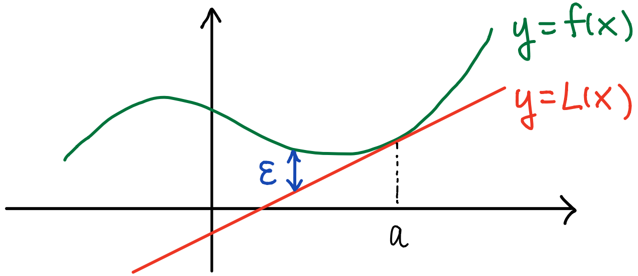
How? In terms of linear approximation and error.

Linear Approximation for $f(x)$

Suppose $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a. Then

$$f(x) \approx L(x) := f(a) + f'(a)(x - a)$$

$L(x)$ is the "best" linear function ($\deg \le 1$ polynomial) to approximate $f(x)$ near a



Tangent at a $y = L(x)$ = "Best" line to approximate the graph $y = f(x)$ near a

Rmk In linear algebra, linear function/map means $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ and $L(c\vec{x}) = cL(\vec{x})$

In particular, $L(\vec{0}) = \vec{0}$. The $L(x)$ defined above may not be linear in this sense.

Error of approximation

$$\varepsilon(x) = f(x) - L(x)$$
$$= f(x) - f(a) - f'(a)\underbrace{(x-a)}_{\Delta x}$$

Note

$$\frac{\varepsilon(x)}{x-a} = \frac{f(x)-f(a)}{x-a} - f'(a)$$

$$\lim_{x\to a} \frac{\varepsilon(x)}{x-a} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} - f'(a)$$
$$= f'(a) - f'(a) = 0$$

Equivalently

$$\lim_{x\to a} \frac{|\varepsilon(x)|}{|x-a|} = 0$$

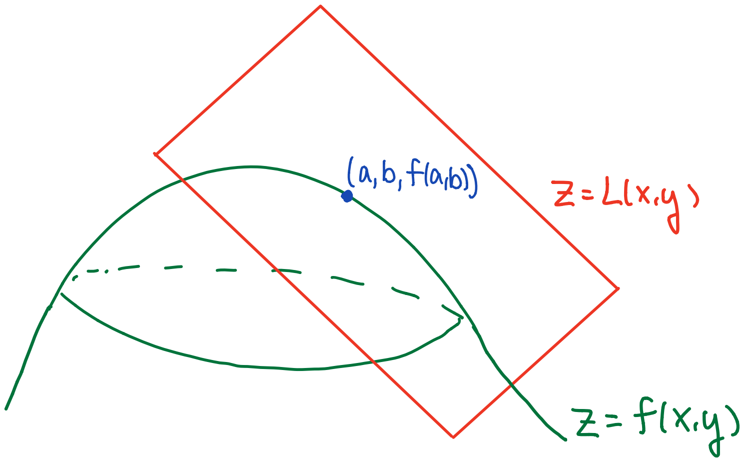
Error is small compared to $\vec{x}-\vec{a}$

In higher dim, graph of $f(\vec{x})$ should be approximated by higher dim linear objects. (eg. Tangent plane of $z=f(x,y)$)

eg Suppose $f:\mathbb{R}^2\to\mathbb{R}$, $f_x(a,b)$, $f_y(a,b)$ exist.

Try to approximate $f(x,y)$ near (a,b):

$$f(x,y) \approx \overbrace{\underbrace{f(a,b)}_{\text{value at }(a,b)} + \underbrace{f_x(a,b)}_{\substack{\text{slope in}\\ x\text{-direction}}}\underbrace{(x-a)}_{\Delta x} + \underbrace{f_y(a,b)}_{\substack{\text{slope in}\\ y\text{-direction}}}\underbrace{(y-b)}_{\Delta y}}^{L(x,y)}$$



Defn Let $\Omega \subseteq \mathbb{R}^n$ be open, $\vec{a} = (a_1, a_2, \cdots, a_n) \in \Omega$

$f: \Omega \to \mathbb{R}$ is said to be **differentiable at $\vec{a}$** if

① All partial derivatives $\frac{\partial f}{\partial x_i}(\vec{a})$ exist for $i = 1, 2, \cdots, n$.

② In the linear approximation for $f(\vec{x})$ at $\vec{a}$,

$$f(\vec{x}) = \underbrace{f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i)}_{L(\vec{x}) = \text{Linear approximation of } f(\vec{x}) \text{ at } \vec{a}} + \underbrace{\varepsilon(\vec{x})}_{\text{error}}$$

the error term $\varepsilon(\vec{x})$ satisfies

$$\lim_{\vec{x} \to \vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0$$

A differentiable function is one which can be well approximated by a linear function locally

Rmk

$$L(\vec{x}) = f(\vec{a}) + \sum_{i=1}^{n} \underbrace{\frac{\partial f}{\partial x_i}(\vec{a})}_{\text{slope of } f \text{ in } x_i\text{-direction at } \vec{a}} \underbrace{(x_i - a_i)}_{\Delta x_i}$$

Note

① $L(\vec{x})$ is a $\deg \leq 1$ polynomial

② $L(\vec{a}) = f(\vec{a})$

③ $\frac{\partial L}{\partial x_i}(\vec{a}) = \frac{\partial f}{\partial x_i}(\vec{a})$

$y = L(\vec{x})$ is a n-plane tangent to $y = f(\vec{x})$ at $\vec{x} = \vec{a}$

eg1 $f(x,y) = x^2 y$

① Show that f is differentiable at $(1,2)$

② Approximate $f(1.1, 1.9)$ using linearization

③ Find tangent plane of $z = f(x,y)$ at $(1,2,2)$

$f(1,2) = 2$

Sol ① $\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2$

$\frac{\partial f}{\partial x}(1,2) = 4 \qquad \frac{\partial f}{\partial y}(1,2) = 1$

$\therefore$ The linearization at $(1,2)$ is

$$L(x,y) = f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2)$$

$$= 2 + 4(x-1) + (y-2)$$

with error term

$$\varepsilon(x,y) = f(x,y) - L(x,y)$$

$$= x^2 y - 2 - 4(x-1) - (y-2)$$

$$\lim_{(x,y)\to(1,2)} \frac{\varepsilon(x,y)}{\|(x,y)-(1,2)\|}$$

$$= \lim_{(x,y)\to(1,2)} \frac{x^2 y - 2 - 4(x-1) - (y-2)}{\sqrt{(x-1)^2 + (y-2)^2}}$$

let $x - 1 = h$
$y - 2 = k$

$$= \lim_{(x,y)\to(1,2)} \frac{(1+h)^2(2+k) - 2 - 4h - k}{\sqrt{h^2 + k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{h^2 k + 2hk + 2h^2}{\sqrt{h^2 + k^2}}$$

let $h = r\cos\theta$
$k = r\sin\theta$

$$= \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta + 2r^2\cos\theta\sin\theta + 2r^2\cos^2\theta}{r}$$

$$= \lim_{r\to 0} r^2\cos^2\theta\sin\theta + 2r\cos\theta\sin\theta + 2r\cos\theta$$

$= 0$ by Sandwich theorem

$\therefore f$ is differentiable at $(1,2)$

② $f(1.1, 1.9) \approx L(1.1, 1.9)$

$$= 2 + 4(1.1-1) + (1.9-2)$$

$$= 2 + 0.4 + (-0.1)$$

$$= 2.3$$

Compare: $f(1.1, 1.9) = 2.299$

③ Tangent at $(1,2,2)$ is

$$z = L(x,y)$$

$$= 2 + 4(x-1) + (y-2)$$

$$z = -4 + 4x + y$$

<u>eg 2</u> Is $f(x,y) = \sqrt{|xy|}$ differentiable at $(0,0)$?

<u>Sol</u> $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$

Similarly $\frac{\partial f}{\partial y}(0,0) = 0$

$$\therefore L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0)$$

$$= 0 + 0 + 0$$

$\therefore L(x,y) \equiv 0$ is the zero function!

Error: $\varepsilon(x,y) = f(x,y) - L(x,y) = \sqrt{|xy|}$

$$\lim_{(x,y)\to(0,0)} \frac{\varepsilon(x,y)}{\|(x,y)-(0,0)\|} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}$$

$$= \lim_{r\to 0} \frac{\sqrt{|r^2\cos\theta\sin\theta|}}{r}$$

$$= \lim_{r\to 0} \sqrt{|\cos\theta\sin\theta|} \quad \text{DNE}$$

different values at $\theta = 0, \frac{\pi}{4}$

$\therefore f$ is not differentiable at $(0,0)$

Rmk In last example, $f(x,y) = \sqrt{|xy|}$, $L(x,y) \equiv 0$

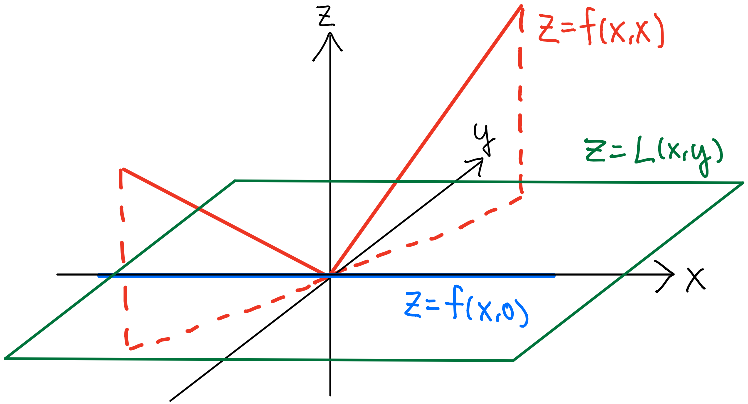
Along the line $y = mx$, $f(x,mx) = \sqrt{|mx^2|} = \sqrt{|m|}\,|x|$

Along x-axis ($m=0$)

$f(x,0) = 0 = L(x,0)$ (Good approximation)

Along $y = x$ ($m=1$)

$f(x,x) = |x|$, $L(x,x) = 0$ (Bad approximation)



In general, our $L(\vec{x})$ is defined using $\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ (Derivatives on coordinate directions)

$$\boxed{\text{Differentiability: Information on coordinate directions } \left(\frac{\partial f}{\partial x_i}\right) \text{ can tell information on every direction}}$$

$\leftarrow$ A strong condition!

Thm If $f(\vec{x})$ is differentiable at $\vec{a}$, then $f(\vec{x})$ is continuous at $\vec{a}$

Pf $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$

L ↑ Linearization of f at $\vec{a}$

f is differentiable at $\vec{a}$

$$\Rightarrow \lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} = 0$$

$$\therefore \lim_{\vec{x}\to\vec{a}} \varepsilon(\vec{x})$$

$$= \lim_{\vec{x}\to\vec{a}} \frac{\varepsilon(\vec{x})}{\|\vec{x}-\vec{a}\|} \cdot \lim_{\vec{x}\to\vec{a}} \|\vec{x}-\vec{a}\|$$

$$= 0 \cdot 0 = 0$$

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x})$$

$$= \lim_{\vec{x}\to\vec{a}} L(\vec{x}) + \lim_{\vec{x}\to\vec{a}} \varepsilon(\vec{x})$$

$$= \lim_{\vec{x}\to\vec{a}} \underbrace{\left(f(\vec{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})(x_i - a_i) \right)}_{\text{polynomial} \Rightarrow \text{continuous}} + 0$$

$$= f(\vec{a})$$

$\therefore$ f is continuous at $\vec{a}$